Chapter 1

Cobordism group Ω_n

Two closed orientable n-manifolds M and N are considered to be the same modulo the cobordism relation if there disjoint union is the boundary of another manifold. That's way studying cobordism theory may be a good way to classify manifolds, and by asking what this have to do with the homology of manifolds. In this projet we will study the case of three dimensional manifolds but before that we will talk the cases $n = 0, 1, 2$.
The main idea in this presentation is to show every closed orientable 3-manifold is the boundary of some orientable 4-manifold by using Dehn surgery on the 3-sphere.

1.1 Preliminaries

Definition 1.1.1. *An n-manifold M will be defined to be a metric space which may be covered by open sets, each of which is homeomorphic with $\mathbb{R}^n$ or the half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$*
M is said to be closed if it is compact and $\partial M = 0$.

Definition 1.1.2. *A handlebody of genus g is the result of attaching g disjoint 1-handles $D^2 \times [-1, 1]$ to a 3-ball B^3 by sewing the parts $D^2 \times \pm 1$ to 2g disjoint disks on the boundary of B^3 in such a way that the result is an orientable 3-manifold with boundary.*

Remark 1.1.1. *Two handlebodies of the same genus are homeomorphic. The boundary of a handlebody of genus g is a closed orientable 2-manifold of the same genus.(see Rolfsen page 239).*

Definition 1.1.3. *A solid torus L is a space homeomorphic with $\mathbb{S}^1 \times D^2$. A framing is a specified homeomorphism $f : S^1 \times D^2 \to L$. By meridian* *we mean a simple closed curve* $\alpha = f(1 \times \partial D^2)$ *and by longitude* $\beta = f(S^1 \times 1)$.

Definition 1.1.4. *Let* L *and* L' *be handlebodies of the same genus,* g *, and let* $f : \partial L' \to \partial L$ *be a homeomorphism. Let* $M = L \bigcup_f L'$*,* M *is a closed orientable 3-manifold and the triple* (L, L', f) *is called Heegaard diagram or Heegaard splitting of genus* g *for* M*.*

Theorem 1.1.1. *Every closed orientable connected 3-manifold has a Heegaard diagram, and hence a well-defined genus.*

Proof. See Rolfsen's book pages 240-241. ■

Definition 1.1.5. *Let* M *be a three dimensional manifold (with boundary) such that there is:*

1. *a link* $L = K_1 \sqcup ... \sqcup K_s$ *of simple closed curves in the interior of* M*,*
2. *disjoint tubular neighborhood* H_i *of* K_i *in the interior of* M*,*
3. *a specified simple closed curve* γ_i *in* ∂H_i *For all* i*.*

Let

$$M' = M - (\mathring{H}_1 \sqcup ... \sqcup \mathring{H}_s) \bigcup_f (H_1 \sqcup ... \sqcup H_s).$$

where f *is a union of homeomorphisms* $f_i : \partial H_i \to \partial H_i$ *, each of which take a meridian curve* α *of* H_i *onto the specified* γ_i*. The 3-manifold* M' *is said to be the result of a Dehn surgery on along the link* L *with surgery instructions (2) and (3).*

Example 1.1.1. *Let* $M = \mathbb{R}^3$ *or* $\mathbb{S}^3$*.*
Let $L = K_1 \sqcup ... \sqcup K_s$ *be an oriented link of simple closed curves in* $\mathbb{R}^3$*. then each component* K_i *has a prefered framing for a tubular neighborhood* H_i *in which the longitude* β_i *is oriented in the same way as* K_i *and the meridian* α_i *has linking number* ± 1 *with* K_i*. Therefore, we may write the curve* γ_i *in terms of the basis:*

$$f_*(\alpha_i) = p_i \beta_i + q_i \alpha_i$$

with ambiguity of a $\pm$ *depending on how one wishes to orient* J_i*. We have that* $q_i = lk(K_i, \gamma_i)$*. The ambiguity disappears if we take the ratio,*

$$r_i = q_i / p_i.$$

The r_i*'s are called surgery coefficients ossociated with the component* K_i*. If* $p_i = 0$ *then* $q_i = \pm 1$ *and we write* $r_i = \infty$*.*

Theorem 1.1.2. *Dehn surgery with coefficients* ± 1 *on the sphere* $\mathbb{S}^3$ *may be view as the result on the boundary of attaching 2-handles to the 4-ball.*

Proof. See Rolfsen pages 261. ■

1.2 Cobordism

Given an oriented manifold M, we will denote by $-M$ the manifold that has the same underlying topological and smooth structure as M, but with the opposite orientation. By ∂M, we mean the boundary of M with the induced orientation. By $M + N$, we mean the disjoint union of M and N; by $M - N$, we mean $M \sqcup$ (-N). By $M = N$, we mean that M is isomorphic to N as oriented manifolds. What we are studying is the equivalence relation of cobordism:

Definition 1.2.1. *Two closed orientable n-manifolds M and N are cobordant if there exists a compact $(n+1)$-manifold with boundary W such that $\partial W = M - N$. We might sometimes write this as $M \sim N$.*

Remark 1.2.1. *This is an equivalence relation. Endeed, It is reflexive since $M - M$ is the boundary of $M \times [0,1]$ as Mark did in the class. It is also symmetric: if $\partial W = M - N$, then $\partial(-W) = N - M$. Finally, we check transitivity. Assume $\partial V = L - M$ and $\partial W = M - N$. Using the collar neighborhood theorem1 , we can define a new manifold X by gluing together the $-M$ component of ∂V and the M component of ∂W. We would then have $\partial X = L - N$.*

We can now define the oriented cobordism groups:

Definition 1.2.2. *The n-th orientable cobordism group Ω_n is the set of closed n-dimensional manifolds together with the group operation + (i.e., disjoint union), modulo the equivalence relation of cobordism.*
To simplify things, we will think of the empty set $\emptyset$ as being an n-manifold for every n. This allows us to set the identity element in Ω_n to be the equivalence class of $\emptyset$. Our notation suggests a natural choice: the inverse of M should be $-M$. And this is indeed the case: $M - M$, as we already mentioned, is the boundary of $M \times [0,1]$, hence is cobordant to $\emptyset$. Note that any manifold M is the boundary of $M \times [0,\infty)$. This is why we should only look at compact manifolds, we would otherwise be studying a completely trivial theory. Since disjoint union is commutative operation, Ω_n is an abelian group. Let us take a look for law dimensional e.g $n = 0, 1, 2$, and 3:

1. *For $n = 0$, a closed orientable 0-mafolds is a finite collection of signed ponits, and that the difference in number between the positive and the negative points determine the cobardism class of manifold. Since a positve point and negative point is the boundary of $[0,1]$, then $\Omega_0 = Z$.*
2. *For $n = 1$, S^1 is the only closed orientable connected n-manifold, and it is the boundary of the disk D^2.*

3. *For $n = 2$, as Mark did in the class, the set of closed oreientable manifolds are, the sphere, torus with one hole, two holes,... And this is the boundaryof a 3-manifold.*

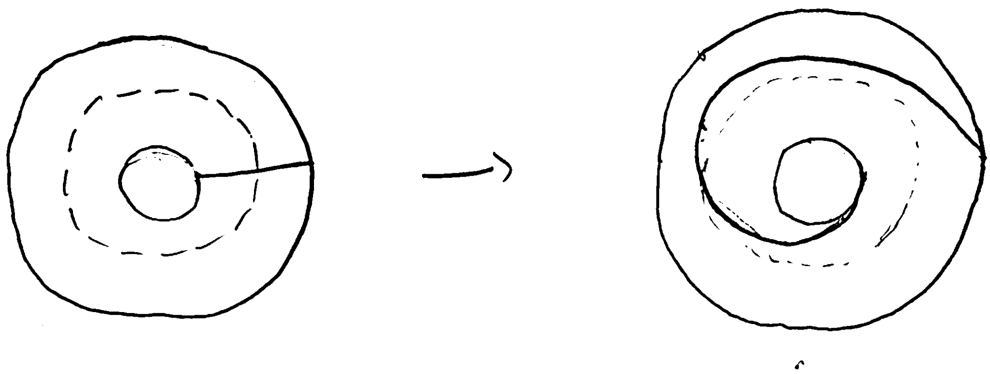
4. *The case $n = 3$ is more harder, and will be studied in the next section.*

1.3 The cobordism group Ω_3

Theorem 1.3.1. *Every closed, orientable 3-manifold is the boundary of some orientable 4-manifold.*

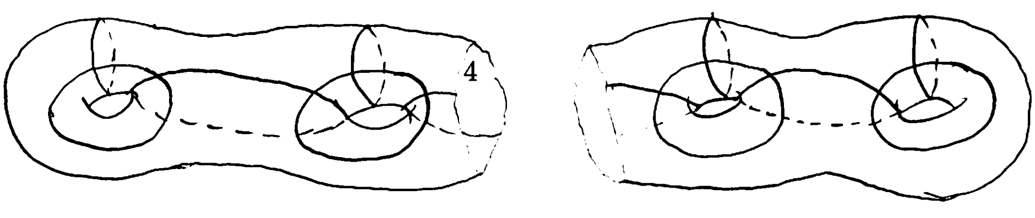
Before to proof this theorem, we will state and proof some lemmas and theorems which will be useful for the proof.

Definition 1.3.1. *Let S be a 2-manifold and $h : S \to S$ a homeomorphism. h is said to be a twist homeomorphism along a curve α on S if h=identity outside an annular neighborhood of α and inside the neighborhood it looks like:*
(see figure).



Theorem 1.3.2. *Let S be a closed orientable surface of genus g . Then every orientation-preserving homeomorphism of S is isotopic to a product of twist homeomorphisms along the $3g - 1$ curves pictured.*
(see figure)

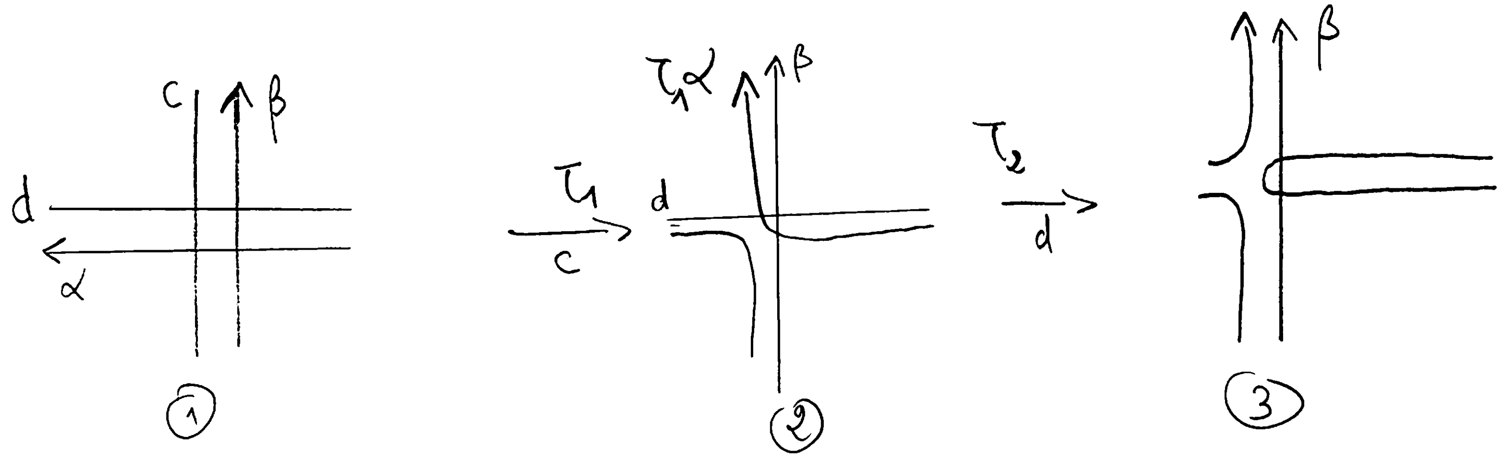
The proof of this theorem will be split has lemma.



Definition 1.3.2. *We say that oriented simple closed curves α and β contained in the interior of the surface S are called twist-equivalent, written $\alpha \sim \beta$, if $h\alpha = \beta$ for some homeomorphism h of S that is in the group of homeomorphisms generated by all twists of S (which includes homeomorphisms isotopic to the identity).*

Lemma 1.3.1. *Let α and β oriented simple closed curves contained in the interior of the surface S, intersect transversely at precisely one point. Then $\alpha \sim \beta$.*

Proof. As shown in the pictures, the first diagram of Figures shows the intersection point of α and β and also a simple closed curve c that runs parallel to, and is slightly displaced from, β. Similarly, d is a slightly displaced copy of α. The second diagram shows $\tau\alpha$, where τ_1 is a twist about c. The third diagram shows $\tau_2\tau_1\alpha$, where τ_2 is a twist about d . In this diagram $\tau_2\tau_1\alpha$ has a doubled-back portion, but we can moved that by a homeomorphism isotopic to the identity to change $\tau_2\tau_1\alpha$ to β. ■



Lemma 1.3.2. *Let α and β oriented simple closed curves contained in the interior of the surface S. Suppose that α and β are disjoint and that neither separates S. Then $\alpha \sim \beta$.*

Proof. Let F be the surface by cutting S along $\alpha \cup \beta$. There is a simple closed curve γ in F that intersects each of α and β transversely at one point. Then, by the previous lemma, $\alpha \sim \gamma$ and $\gamma \sim \beta$. Therefore $\alpha \sim \beta$. ■

Lemma 1.3.3. *Let α and β oriented simple closed curves contained in the interior of the surface S, and that neither separates S. Then $\alpha \sim \beta$.*

Proof. The proof of this lemma can be done by using the two first lemmas. For more detail see the Lickorish book pages 126-127.
■

Corollary 1.3.1. *Let $\alpha_1, \cdots, \alpha_n$ be disjoint simple closed curves in the interior of S the union ofwhich does not separate S. Let $\beta_1, \cdots, \beta_n$ be disjoint simple closed curves in the interior of S the union ofwhich does not separate S. Then there is a homeomorphism h of S that is in the group generated by twists, so that $h\alpha_j = \beta_j$ .for each $j = 1, \cdots, n$,*

Proof. We going to show this by induction. By the previous lemma, there is a twist homeomorphism τ_1 which send α_1 to β_1. Now, assume there Is homeomorphism f which send $\alpha_1, \cdots, \alpha_{n-1}$ to $\beta_1, \cdots, \beta_{n-1}$. Again using the previous lemma, there is a twist homeomorphism τ_n which send α_n to β_n. This $h = \tau_n \circ f$.

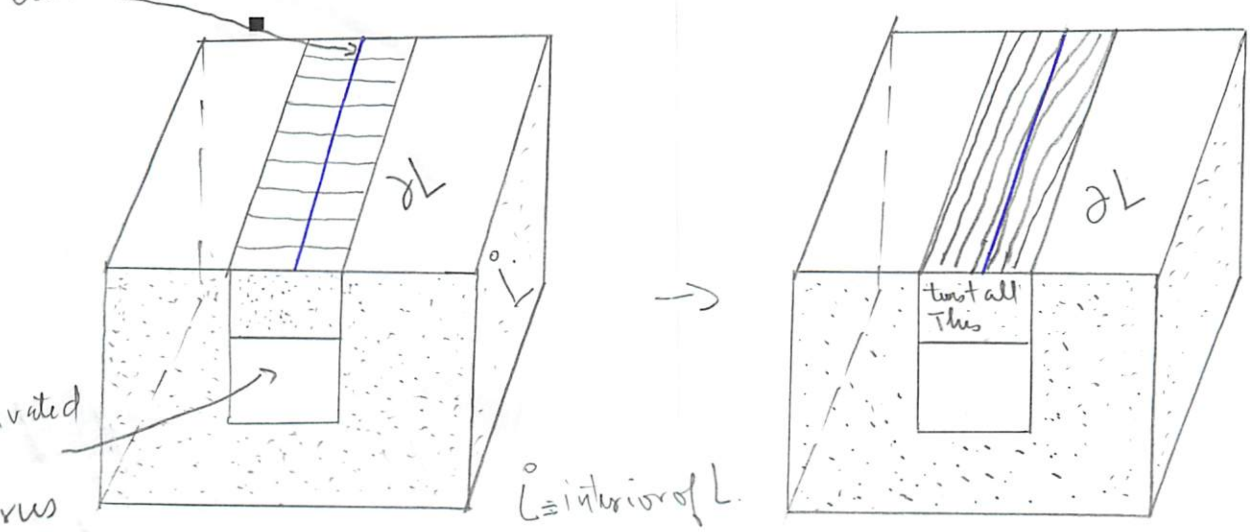
■ This complete the proof of the Lickorish theorem.

Lemma 1.3.4. *Let L and L' are handlebodies of the same genus and $f : \partial L \to \partial L'$ be any homeomorphism. Then there exist disjoint solid tori $H_1, ..., H_s$ in L and $H'_1, ..., H'_s$ in L' such that f extends to a homeomorphism $\overline{f} : L - (\mathring{H}_1 \sqcup ... \sqcup \mathring{H}_s) \to L' - (\mathring{H}'_1 \sqcup ... \sqcup \mathring{H}'_s)$.*

Proof. Since L and L' have same genus, then they are homeomorphic, therefore we may assume $L = L'$, and $f : \partial L \to \partial L$ preserves orientation. We know that any homeomorphism of ∂L which is isotopic to the identity can be extends to a homeomorphism of all L to it self, by moving only a collar of the boundary. So by the the previous theorem we can write $f = g_1, ..., g_s$, a composition of twists a long some or all $3g - 1$ curves. We have that g_1 is the identity off an annular neighborhood V of its twisting curve in ∂L. Consider a tunnel excavated from L just under this annulus, within a collar of ∂L(see figure). This tunnelis a solid torus, call it H_1. The region between H_1 an V is a copy of $V \times [0, 1]$, which may be twisted by $g_1 \times id$. Therefore g_1 can be extends by this map, together with the identity elsewhere on $L - \mathring{H}_1$. Let call this extension $\overline{g_1} : L - \mathring{H}_1 \to L - \mathring{H}_1$. Similarly g_2 may be extended to a homeomorphism $\overline{g_2} : L - \mathring{H}_2 \to L - \mathring{H}_2$.
By excavated slightly deeper than before, if necessary, it can be arrange that H_2 missed $H - 1$ and that $\overline{g_1}$ is the identity on H_2. Inductively, we define in this way a collection of disjoint solid torus tunnels $H_1, ..., H_s$ and extensions $\overline{g_i} : L - \mathring{H}_i \to L - \mathring{H}_i$ so that $\overline{g_i}$ is fixed on H_j for $i < j$.
Let $\overline{f}$ be the compositions $\overline{f} = \overline{g_s} ... \overline{g_2 g_1}$, restricted to $L - (\mathring{H}_1, ..., \mathring{H}_s)$. The solid tori to be deleted to get the range of $\overline{f}$ are $H'_s = H_s$ and $H'_i =$

$g_s...g_{i+1}(H_i)$ for $i < s$. Finally we get,

$$\overline{f} : L - (\mathring{H}_1 \sqcup ... \sqcup \mathring{H}_s) \to L - (\mathring{H}'_1 \sqcup ... \sqcup \mathring{H}'_s).$$



Lemma 1.3.5. *Every closed orientable, connected 3-manifold may be obtained by Surgery on a link in $\mathbb{S}^3$. Moreover, one may always find such a Surgery presentation in which the Surgery coefficients are all ± 1 and the individual components of the link are unknotted.*

Proof. Let M be a closed, connected, orientable 3-manifold. By theorem 1.2, one can choose Heegaard decomposition of the same genus:

$$\mathbb{S}^3 = L \sqcup_f L'$$

and

$$M = N \sqcup_{f'} N'$$

where $f : \partial L' \to \partial L$ $f' : \partial N' \to \partial N$ are homeomorphisms attaching the handlebodies. Since handlebodies of the same genus are homeomorphic, let $g : L \to N$ be a homeomorphism. We have,

$$h = (f')^{-1} \circ g \circ f : \partial L' \to \partial N'$$

is a homeomorphism. Therefore, by the lemma 3.1 h can be extends to a homeomorphism

$$\overline{h} : L' - (\mathring{H}_1 \sqcup ... \sqcup \mathring{H}_s) \to N' - (\mathring{H}'_1 \sqcup ... \sqcup \mathring{H}'_s)$$

where the H_i's and H'_j's are disjoint solid tori.
This homeomorphism extends to a homeomorphism

$$\overline{h'} : \mathbb{S}^3 - (\mathring{H}_1 \sqcup ... \sqcup \mathring{H}_s) \to M - (\mathring{H}'_1 \sqcup ... \sqcup \mathring{H}'_s).$$

In the from of the Lemme 3.1, we have seen that h' carries ∂H_i to $\partial H'_i$ for all i and that the preimage of a meridian of H'_i is a meridian $\pm$ longitude of H_i. Thus, M is the result of a surgery of $\mathbb{S}^3$ with coefficients ± 1 on the solid tori $H_1, ..., H_s$.

■

Proof. (theorem 3.1) Without loss of generality we can assume M is connected. By the Lemma 3.2 M is the result of a surgery of $\mathbb{S}^3$ with coefficients ± 1, and by the theorem 1.3 this is the result on the boundary of attaching 2-handles to the 4-ball. ■

1.4 Conclusion

$\Omega_3 = 0$. This is perhaps the nicest and most direct proof but there were previous proofs, Rochlin, Thom.